

**Massachusetts Institute of Technology**  
**Department of Electrical Engineering and Computer Science**  
6.685 Electric Machinery

Class Notes 5: Winding Inductances  
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## 1 Introduction

The purpose of this document is to show how the inductances of windings in round- rotor machines with narrow air gaps may be calculated. We deal only with the idealized air- gap magnetic fields, and do not consider slot, end winding, peripheral or skew reactances. We do, however, consider the space harmonics of winding magneto-motive force (MMF).

## 2 Description of Stators

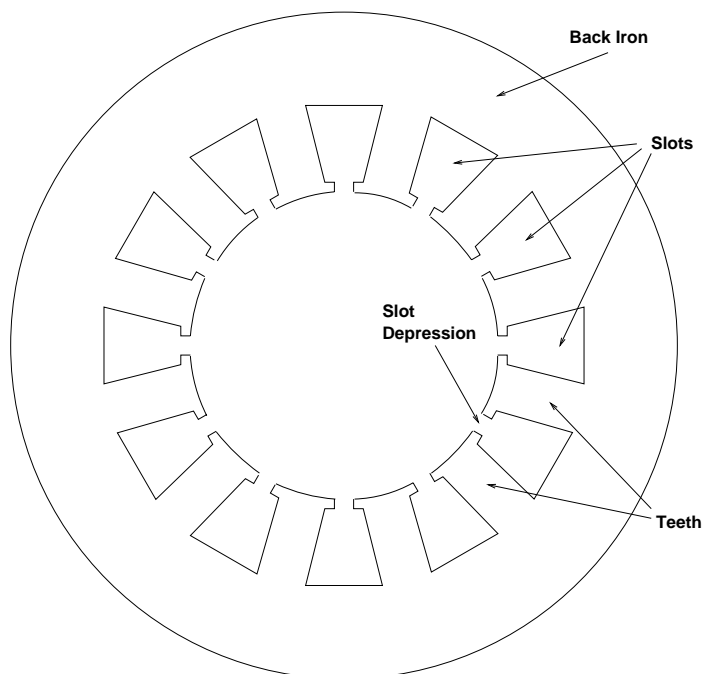


Figure 1: Stator Cross-Section

Figure 1 shows a cartoon view of an axial cross-section of a twelve-slot stator. Actually, what is shown is the shape of a thin sheet of steel, or *lamination* that is used to make up the magnetic circuit. The iron is made of thin sheets to control eddy current losses. Thickness varies according to frequency of operation, but in machines for 60 Hz (the vast bulk of machines made for industrial

use), lamination thickness is typically .014" (.355 mm). These are stacked to make the magnetic circuit of the appropriate length. Windings are carried in the slots of this structure.

Figure 1 shows trapezoidal slots with teeth of approximately uniform cross-section over most of their length but wider extent near the air-gap. The tooth ends, in combination with the relatively narrow slot depression region, help control certain parasitic losses in the rotor of many machines by improving uniformity of the air-gap fields, increase the air-gap permeance and help hold the windings in the slots. It should be noted that large machines, with what are called "form wound" coils, have straight-sided rectangular slots and consequently teeth of non-uniform cross-section. The description that follows will hold for both types of machine.

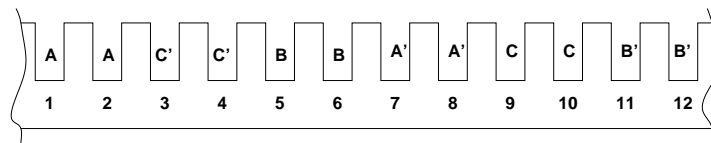


Figure 2: Full-Pitched Winding

To simplify the discussion, imagine the slot/tooth region to be "straightened out" as shown in Figure 2. This shows a three-phase, two-pole winding in the twelve slots. Such a winding would have two slots per pole per phase. One of the two coils of phase A would be wound in slots 1 and 7 (six slots apart).

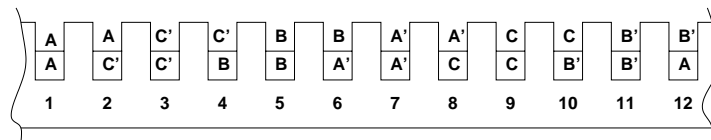


Figure 3: Five-Sixths-Pitched Winding

Machines are seldom wound as shown in Figure 2 for a variety of reasons. It is usually advantageous in reducing the length of the end turns and to reducing space harmonic effects in the machine (usually bad effects!) to wind the machine with "short-pitched" windings as shown in Figure 3. Each phase in this case consists of four coils (two per slot). The four coils of Phase A would span between slots 1 and 6, slots 2 and 7, slots 7 and 12 and slots 8 and 1. Each of these coil spans is five slots, so this choice of winding pattern is referred to as "Five-Sixths" pitch.

So this cartoon-figure machine stator (which could represent either a synchronous or induction motor or generator) has both *breadth* because there are more than one slots per pole per phase, and it may have the need for accounting for winding *pitch*. What follows in this note is a simple protocol for estimating the important air-gap fields and inductances.

### 3 Winding MMF

To start, consider the MMF of a full- pitch, concentrated winding as shown in schematic form in Figure 4. Assuming that the winding has a total of  $N$  turns over  $p$  pole- pairs, and is carrying current  $I$  the MMF is:

$$F = - \sum_{\substack{n=1 \\ \text{nodd}}}^{\infty} \frac{4}{n\pi} \frac{NI}{2p} \sin np\theta \quad (1)$$

This distribution is shown, as a function of angle  $\theta$  in Figure 5.

This leads directly to magnetic flux density in the air- gap:

$$B_r = - \sum_{\substack{n=1 \\ \text{nodd}}}^{\infty} \frac{\mu_0}{g} \frac{4}{n\pi} \frac{NI}{2p} \sin np\theta \quad (2)$$

Note that a real winding, which will most likely not be full- pitched and concentrated, will have a *winding factor* which is the product of pitch and breadth factors, to be discussed later.

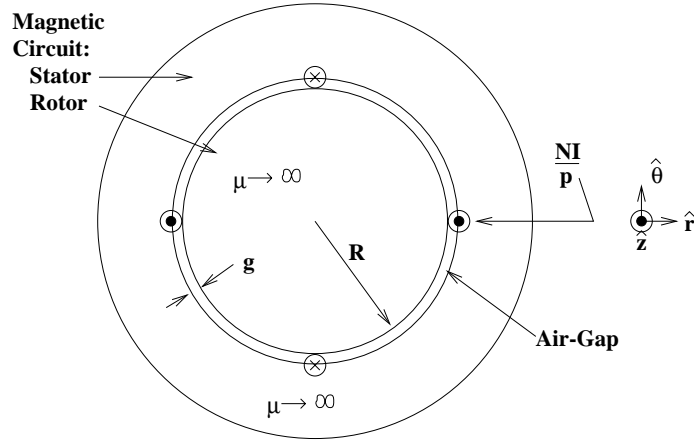


Figure 4: Primitive Geometry Problem

Now, suppose that there is a polyphase winding, consisting of more than one phase (we will use three phases), driven with one of two types of current. The first of these is *balanced*, current:

$$\begin{aligned} I_a &= I \cos(\omega t) \\ I_b &= I \cos(\omega t - \frac{2\pi}{3}) \\ I_c &= I \cos(\omega t + \frac{2\pi}{3}) \end{aligned} \quad (3)$$

Conversely, we might consider *Zero Sequence* currents:

$$I_a = I_b = I_c = I \cos \omega t \quad (4)$$

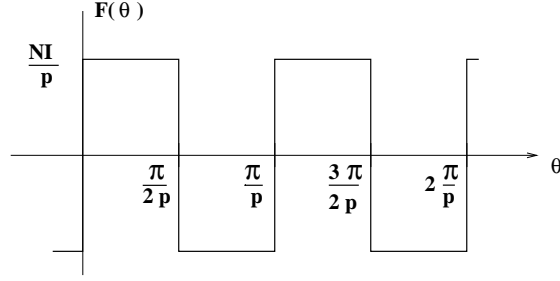


Figure 5: Air-Gap MMF

Then it is possible to express magnetic flux density for the two distinct cases. For the *balanced* case:

$$B_r = \sum_{n=1}^{\infty} B_{rn} \sin(np\theta \mp \omega t) \quad (5)$$

where

- The upper sign holds for  $n = 1, 7, \dots$
- The lower sign holds for  $n = 5, 11, \dots$
- all other terms are zero

and

$$B_{rn} = \frac{3}{2} \frac{\mu_0}{g} \frac{4}{n\pi} \frac{NI}{2p} \quad (6)$$

The zero- sequence case is simpler: it is nonzero only for the *triplen* harmonics:

$$B_r = \sum_{n=3,9,\dots}^{\infty} \frac{\mu_0}{g} \frac{4}{n\pi} \frac{NI}{2p} \frac{3}{2} (\sin(np\theta - \omega t) + \sin(np\theta + \omega t)) \quad (7)$$

Next, consider the flux from a winding on the rotor: that will have the same form as the flux produced by a single armature winding, but will be referred to the rotor position:

$$B_{rf} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\mu_0}{g} \frac{4}{n\pi} \frac{NI}{2p} \sin np\theta' \quad (8)$$

which is, substituting  $\theta' = \theta - \frac{\omega t}{p}$ ,

$$B_{rf} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\mu_0}{g} \frac{4}{n\pi} \frac{NI}{2p} \sin n(p\theta - \omega t) \quad (9)$$

The next step here is to find the flux linked if we have some air- gap flux density of the form:

$$B_r = \sum_{n=1}^{\infty} B_{rn} \sin(np\theta \pm \omega t) \quad (10)$$

Now, it is possible to calculate flux linked by a single-turn, full-pitched winding by:

$$\Phi = \int_0^{\frac{\pi}{p}} B_r R l d\theta \quad (11)$$

and, using (10), this is:

$$\Phi = 2Rl \sum_{n=1}^{\infty} \frac{B_{rn}}{np} \cos(\omega t) \quad (12)$$

This allows us to compute self- and mutual- inductances, since winding flux is:

$$\lambda = N\Phi \quad (13)$$

The end of this is a set of expressions for various inductances. It should be noted that, in the real world, most windings are not full-pitched nor concentrated. Fortunately, these shortcomings can be accommodated by the use of *winding factors*.

The simplest and perhaps best definition of a winding factor is the ratio of flux linked by an actual winding to flux that would have been linked by a full- pitch, concentrated winding with the same number of turns. That is:

$$k_w = \frac{\lambda_{actual}}{\lambda_{full-pitch}} \quad (14)$$

It is relatively easy to show, using reciprocity arguments, that the winding factors are also the ratio of effective MMF produced by an actual winding to the MMF that would have been produced by the same winding were it to be full- pitched and concentrated. The argument goes as follows: mutual inductance between any pair of windings is reciprocal. That is, if the windings are designated *one* and *two*, the mutual inductance is flux induced in winding *one* by current in winding *two*, and it is also flux induced in winding *two* by current in winding *one*. Since each winding has a winding factor that influences its linking flux, and since the mutual inductance must be reciprocal, the same winding factor must influence the MMF produced by the winding.

The winding factors are often expressed for each space harmonic, although sometimes when a winding factor is referred to without reference to a harmonic number, what is meant is the space factor for the space fundamental.

Two winding factors are commonly specified for ordinary, regular windings. These are usually called *pitch* and *breadth* factors, reflecting the fact that often windings are not *full pitched*, which means that individual turns do not span a full  $\pi$  electrical radians and that the windings occupy a range or breadth of slots within a phase belt. The breadth factors are ratios of flux linked by a given winding to the flux that would be linked by that winding were it full- pitched and concentrated. These two winding factors are discussed in a little more detail below. What is interesting to note, although we do not prove it here, is that the winding factor of any given winding is the *product* of the pitch and breadth factors:

$$k_w = k_p k_b \quad (15)$$

With winding factors as defined by (14) and the sections below, it is possible to define winding inductances. For example, the *synchronous* inductance of a winding will be the apparent inductance of one phase when the polyphase winding is driven by a *balanced* set of currents as in (3). This is, approximately:

$$L_d = \sum_{n=1,5,7,\dots}^{\infty} \frac{3}{2} \frac{4}{\pi} \frac{\mu_0 N^2 R l k_{wn}^2}{p^2 g n^2} \quad (16)$$

This expression is approximate because it ignores the asynchronous interactions between higher order harmonics and the rotor of the machine. These are beyond the scope of this note.

Zero- sequence inductance is the ratio of flux to current if a winding is excited by zero sequence currents, as in (4):

$$L_0 = \sum_{n=3,9,\dots}^{\infty} 3 \frac{4}{\pi} \frac{\mu_0 N^2 R l k_{wn}^2}{p^2 g n^2} \quad (17)$$

And then mutual inductance, as between a *field* winding (*f*) and an *armature* winding (*a*), is:

$$M(\phi) = \sum_{\substack{n=1 \\ \text{nodd}}}^{\infty} \frac{4}{\pi} \frac{\mu_0 N_f N_a k_{fn} k_{an} R l}{p^2 g n^2} \cos(np\phi) \quad (18)$$

## 4 Winding Factors

Now we turn our attention to computing the winding factors for simple, regular winding patterns. We do not prove but only state that the winding factor can, for regular winding patterns, be expressed as the product of a *pitch* factor and a *breadth* factor, each of which can be estimated separately.

### 4.1 Pitch Factor

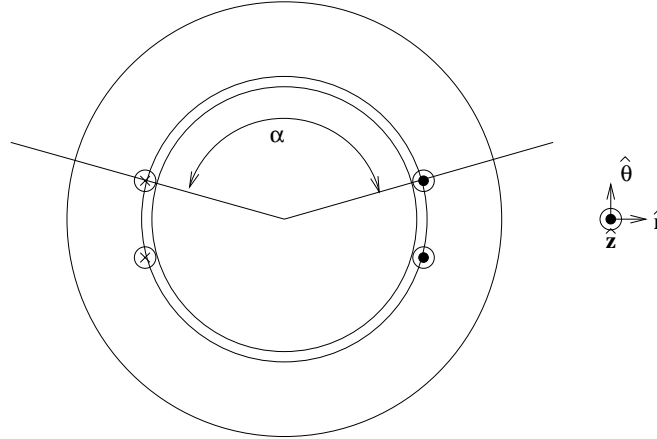


Figure 6: Short-Pitched Coils

Pitch factor is found by considering the flux linked by a less- than- full pitched winding. Consider the situation in which radial magnetic flux density is:

$$B_r = B_n \sin(np\theta - \omega t) \quad (19)$$

A winding with pitch  $\alpha$  will link flux (see Figure 6:

$$\lambda = N l \int_{\frac{\pi}{2p} - \frac{\alpha}{2p}}^{\frac{\pi}{2p} + \frac{\alpha}{2p}} B_n \sin(np\theta - \omega t) R d\theta \quad (20)$$

Pitch  $\alpha$  refers to the angular displacement between sides of the coil, expressed in *electrical* radians. For a full- pitch coil  $\alpha = \pi$ .

The flux linked is:

$$\lambda = \frac{2NlRB_n}{np} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\alpha}{2}\right) \quad (21)$$

Using the definition (14), the *pitch* factor is seen to be:

$$k_{pn} = \sin \frac{n\alpha}{2} \quad (22)$$

## 4.2 Breadth Factor

Now for *breadth* factor. This describes the fact that a winding may consist of a number of coils, each linking flux slightly out of phase with the others. A regular winding will have a number (say  $m$ ) coil elements, separated by *electrical* angle  $\gamma$ . (See Figure 7

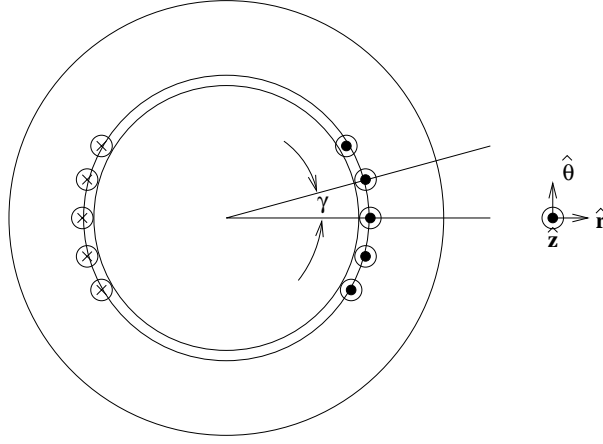


Figure 7: Distributed Coils

A full- pitch coil with one side at angle  $\xi$  will, in the presence of magnetic flux density as described by (19), link flux:

$$\lambda = Nl \int_{\frac{\xi}{p}}^{\frac{\pi}{p} - \frac{\xi}{p}} B_n \sin(np\theta - \omega t) R d\theta \quad (23)$$

This is readily evaluated to be:

$$\lambda = \frac{2NlRB_n}{np} \text{Re} \left( e^{j(\omega t - n\xi)} \right) \quad (24)$$

where in (24), complex number notation has been used for convenience in carrying out the rest of this derivation.

What happens here is that the coils link fluxes that differ in *phase*, so the addition of flux is as shown in vector form in Figure 8.



Figure 8: Vector Flux Addition

Now: if the winding is distributed into  $m$  sets of slots and the slots are evenly spaced, the angular position of each slot will be:

$$\xi_i = i\gamma - \frac{m-1}{2}\gamma \quad (25)$$

and the number of turns in each slot will be  $\frac{N}{mp}$ , so that actual flux linked will be:

$$\lambda = \frac{2NlRB_n}{np} \frac{1}{m} \sum_{i=0}^{m-1} \text{Re} \left( e^{j(\omega t - n\xi_i)} \right) \quad (26)$$

The *breadth* factor is then simply:

$$k_b = \frac{1}{m} \sum_{i=0}^{m-1} e^{-jn(i\gamma - \frac{m-1}{2}\gamma)} \quad (27)$$

Note that (27) can be written as:

$$k_b = \frac{e^{jn\gamma \frac{m-1}{2}}}{m} \sum_{i=0}^{m-1} e^{-jni\gamma} \quad (28)$$

Now, focus on that sum. We know that any covering geometric sum has a simple sum:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad (29)$$

and that a truncated sum is:

$$\sum_{i=0}^{m-1} = \sum_{i=0}^{\infty} - \sum_{i=m}^{\infty} \quad (30)$$

Then the sum in (28) can be written as:

$$\sum_{i=0}^{m-1} e^{-jni\gamma} = (1 - e^{jnm\gamma}) \sum_{i=0}^{\infty} e^{-jni\gamma} = \frac{1 - e^{jnm\gamma}}{1 - e^{-jn\gamma}} \quad (31)$$

Now, inserting the results of (31) into (28), and using the definitions for sine, the breadth factor is found:

$$k_{bn} = \frac{\sin \frac{nm\gamma}{2}}{m \sin \frac{n\gamma}{2}} \quad (32)$$



### 4.3 Alternate Derivation of Breadth Factor

Most textbooks, if they bother to prove the Breadth Factor, use a geometric proof as shown in Figure 9.

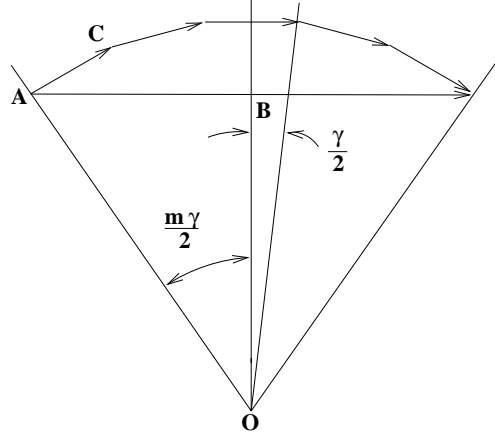


Figure 9: Alternate Proof of Breadth Factor

The short vectors (e.g. AC) represent the voltages induced in individual coils. In fact, what is shown in this figure is the same as is shown in Figure 8, but spread out to show the actual addition. Now, note that if each of the vectors is bisected by a line segment at right angles, all of those line segments meet at point O. The line segment that includes OB is one of these. Line segments that run from O to the ends of the vectors will have an angle  $\frac{\gamma}{2}$  from the bisectors of the vectors. Similarly, the line segment OA has an angle of  $\frac{m\gamma}{2}$  with respect to the bisector of the resultant voltage vector.

Now, if we note  $F_1$  as the length of each of the individual coil voltage vectors and  $F$  as the length of the resultant sum, the length of half of the bisector is:

$$AB = \frac{F}{2} = OA \sin \frac{m\gamma}{2} \quad (33)$$

but then

$$\frac{1}{2}AC = \frac{1}{2}F_1 = OA \sin \frac{\gamma}{2} \quad (34)$$

Then the resultant vector is:

$$F = 2AB = mF_1 \frac{\sin m\frac{\gamma}{2}}{m \sin \frac{\gamma}{2}} \quad (35)$$